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APPLICATION OF DIMENSIONAL ANALYSIS TO THE PREDICTION OF MECHANICAL RELIABILITY

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This report documents a study to determine the feasibility of applying dimensional analysis to the prediction of mechanical reliability. The study concludes that dimensional analysis is an essential tool for the development of empirical models for mechanical reliability. As an example, dimensional analysis is applied to the prediction of mean-time-to failure for involute splines. Advantages of dimensional analysis are enumerated.

ABSTRACT

This report documents a study to determine the feasibility of applying dimensional analysis to the prediction of mechanical reliability. The study concludes that dimensional analysis is an essential tool for the development of empirical models for mechanical reliability. As an example, dimensional analysis is applied to the prediction of mean-time-to failure for involute splines. Advantages of dimensional analysis are enumerated.

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INTRODUCTION

In conjunction with research in "Analysis of Mechanical Reliability Prediction Techniques" being performed by the Naval Weapons Engineering Support Activity (ESA-11), the ultimate goal of which is the publication of a mechanical reliability design guide, Louisiana Tech University has performed studies aimed at improving mechanical reliability estimation and prediction techniques.

The objective of the research on which this report is based was to demonstrate that dimensional analysis is an essential tool for the development of empirical mathematical models for mechanical reliability. While dimensional analysis by itself cannot produce a mathematical model, dimensional analysis reduces the quantity of data required and makes the resulting model simpler and more general. Dimensional analysis also simplifies the application of regression analysis in the development of mathematical models for mechanical reliability prediction. Furthermore, the empirical approach using dimensional analysis does not require a theoretical understanding of the inner workings of the phenomenon being modeled.

DIMENSIONAL ANALYSIS

Dimensional analysis has been used extensively in the fields of convection heat transfer and fluid mechanics to reduce the quantity of experimental data and theoretical knowledge required for the development of empirical mathematical models. Applications of dimensional analysis may be found in almost every text on these subjects. References (1)* and (2) are examples of such texts. The application of dimensionless groupings of variables produced by application of dimensional analysis is used universally in developing empirical mathematical models (mathematical models based directly on experimental data rather than theory) in these fields. Dimensionless groupings of variables such as the Reynolds Number, Prandtl Number, friction factor, Mach Number and Nusselt Number are familiar to all who work in these areas.

Dimensional analysis has also been applied to electrical circuits (3, 4, 5, 6), electromagnetics (4, 7), electronics (4), mechanics (3, 4, 7, 8), stress analysis (3, 4, 5, 7), celestial mechanics (8), geology (4), thermodynamics (3, 4, 7), radiation heat transfer (4, 5), and conduction heat transfer (3, 7).

Although the reduction in experimental data required and the elimination of the need for detailed theoretical knowledge of the inner mechanisms of the phenomenon are exactly what is needed for mechanical reliability, workers in the field of mechanical reliability seem to be

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*Numbers in parentheses refer to an entry in the List of References.

unaware of the tremendous potential of dimensional analysis. The only application of dimensional analysis to mechanical reliability that has been found thus far is incomplete and was not used to exploit the advantages of the method at all (9). Reference (10) cites as a guideline for parameter selection in using regression analysis to predict reliability of mechanical equipments, "meaningful parameter combinations will be made so that the number of parameters can be reduced with little or no information loss." No significant actual use of "parameter combinations" is reported in reference (10), however.

Principles of Dimensional Analysis

Dimensional analysis is a tool that can be used along with data to develop an empirical mathematical model of a physical phenomenon. The use of dimensional analysis is based on ". . . the single premise *that the phenomenon can be described by a dimensionally* correct equation among certain variables." (7) It is important to note that we do not have to be able to write the equation at the outset in order to apply dimensional analysis. Indeed, if we were able to write the equation (mathematical model), we would not need to employ dimensional analysis. We merely need the assurance that the physical phenomenon could be expressed by an equation. One of the main advantages of dimensional analysis is the fact that dimensional analysis does not require that we have a knowledge of the inner mechanisms of a physical phenomenon.

In order to apply dimensional analysis, we do need to know enough about the physical phenomenon to identify all the variables that affect the phenomenon. If a variable that affects the phenomenon is omitted,

the resulting dimensionless variables obviously cannot be used to accurately model the phenomenon. Once the pertinent variables are selected, dimensional analysis is used to group the variables into dimensionless combinations of these variables. Thus, dimensional analysis reduces the number of variables that describes a physical phenomenon.

Dimensions and Units

Before proceeding further, the difference between units and dimensions must be made clear. Depending on which system of dimensions is used, the dimension of a physical variable may be expressed in terms of mass [M]*, length [L], time [T] and temperature [θ] (mass system); force [F], length [L], time [T] and temperature [θ](force system), etc. It can be shown that the different systems are equivalent. In this report, the mass system (sometimes called the MLTθ system) will be used exclusively. A physical variable has but one set of dimensions; however, its magnitudes may be expressed in a number of different units. For example, the diameter of a cylinder has the dimension length [L], but the numerical value of the diameter may be expressed in a number of different units such as millimeters, centimeters, inches, feet, etc. Likewise, shear stress has the dimension $[ML^{-1}T^{-2}]$ but its numerical values may be expressed in pounds per square inch, pounds per square foot, newtons per square millimeter, etc.

Dimensional analysis (a) predicts the number of independent dimensionless groups that can be formed from a group of physical variables and (b) provides a systematic means of forming the groups. Dimensional

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*The symbols for basic dimensions are enclosed in brackets.

analysis does not provide the functional relationship between the dimensionless variables (mathematical model)--this relationship must be determined by some other means, usually from experimental data. If experimental or other numerical data is available, regression analysis can be employed to determine an equation relating the dimensionless variables.

Advantages of Dimensional Analysis

The important advantages of dimensional analysis are summarized as follows:

1. The number of variables required to describe a physical phenomenon is reduced. The number of dimensionless variables is given by Van Driest (11) as:

> The number of dimensionless products in a complete set is equal to the total number of variables minus the maximum number of these variables that will not form a dimensionless product.

A rule that is equivalent and easier to apply is given by Langhaar (7):

> The number of dimensionless products in a complete set is equal to the total number of variables minus the rank of their dimensional matrix.

In most instances, the "maximum number of these variables that will not form a dimensionless product" or the "rank of their dimensional matrix" is just the number of primary dimensions required to express the dimensional formulas of the variables involved. For example, if a physical phenomenon is described by 7 variables and the dimensional formulas of

these variables can be expressed using the 3 primary dimensions [M], [L], and [T], then the physical phenomenon will be described by 7-3 = 4 dimensionless variables.

Corollary advantages resulting from the reduction in the number of variables are:

a. Data requirements are greatly reduced.

b. The fitting of equations to data is facilitated.

c. The resulting mathematical models are easier to use and to present.

To illustrate these advantages, consider again a physical phenomenon that is described by 7 variables. We wish to establish a relationship (model) that predicts the dependent variable in terms of the 6 independent variables. If we use experimental data to determine the relationship by obtaining experimentally a value for the dependent variable for 5 values of each of the independent variables, then 5^6 = 15,625 data points are required. If the results are presented in graphical form, with each chart representing the dependent variable as a function of 2 independent variables, the remaining 4 independent variables being constant for that particular chart, then an unwieldy 5^4 = 625 charts are required. If the relationship is expressed in equation form, then an equation containing 6 independent variables must be fitted to the data. The latter is a formidable task since the relations are most often non-linear and many different variable transformations must be tried for each variable in order to obtain an equation that fits the data well.

If dimensional analysis is used to reduce the number of variables from 7 to 4 and experimental data is taken for 5 values of each of the

3 independent variables, then only $5^3 = 125$ data points are required. In this case data requirements are reduced by over 99 percent through the use of dimensional anslysis! Additionally, the results can be presented graphically by a set of 5 charts rather than 625. If an equation is to be fitted to the data, the equation would contain only 3 independent variables; consequently, the equation would be easier to obtain and simpler to use than an equation containing 6 independent variables.

2. A knowledge of the inner mechanism of the phenomenon is not required.

3. The model that results from the utilization of dimensionless groupings is very general in nature. For example, a particular value of a dimensionless grouping of 4 variables may result from any one of an infinite number of combinations of values of the 4 variables. This characteristic of dimensional analysis will be discussed in a subsequent example.

4. The value of a dimensionless grouping of variables can be varied in an experiment by varying the value of only one of the variables. The experiment can therefore be planned so that the value of the dimensionless grouping is controlled by varying the variable that is most easily and economically controlled.

5. Dimensionless groupings can be interpreted physically, thus adding to the understanding of the phenomenon.

Characteristics of Dimensional Analysis

Some important characteristics of dimensional analysis may be summarized as follows:

1. In order to successfully apply dimensional analysis, the phenomenon must be understood well enough so that all the variables that affect the phenomenon are identified.

2. All variables must be quantifiable.

3. Dimensional analysis by itself does not provide a model of the phenomenon. Information concerning the relationship between the variables, most often in the form of experimental data, is required.

4. Dimensional analysis does not explain the inner mechanism of a phenomenon.

5. The more we know about the phenomenon, the easier the dimensional analysis and curve fitting procedures.

APPLICATION OF DIMENSIONAL ANALYSIS TO A TYPICAL CASE: SPLINE RELIABILITY PREDICTION

The use of dimensionless ratios as a tool in mechanical reliability has potential for a wide range of components and systems. For illustration purposes a single component with simplified assumptions will be used in the following paragraphs. The class of mechanical equipment which fails mainly due to wear in constant rotation has a large number of variables suitable for dimensionless grouping and a method suitable for one component could be expected to be similar to a method suitable for another component. Components in this class include gears, cams, splines, sprockets, bearings, seals, impellers, rollers, casters and tires.

Since splines are of special interest in mechanical reliability studies by the Navy at the present time they were chosen as an example to illustrate the method of dimensionless ratios. Splines may be classified as fixed or working, with tooth types of different geometries (such as full depth involute, half depth involute, circular, etc.) and possible failure modes include fatigue, shear failure, and fretting corrosion as well as the primary failure mode--wear.

Forming the Dimensionless Groups

A listing of the more common independent factors affecting the wear life of splines with a specified type of teeth includes: length, diameter, torque, speed, tooth hardness, and misalignment. Other factors

such as contact area, bearing stress, and horsepower transmitted are dependent upon the above factors and are not included as variables.

In order to be expressible in equation form, a variable must be quantifiable. The present state of knowledge for splines does not permit the effect of lubrication on reliability to be quantified. Similarly, shock loading effects and tooth type have not been quantified. It should be noted that shock loading effects can be quantified; however, much experimental work would be required and a better solution would be to have a separate mathematical model for each application (general type of shock effect). In addition, the best approach is a separate mathematical model for each type of non-quantifiable parameter, i.e., a different model for each type of lubrication and tooth type.

Adding a dependent variable, mean-time-to failure, to the list of six independent variables gives seven variables. A letter symbol will be used for each as follows:

t -- Torque, inch-pounds

N -- Speed, revolutions per minute

ℓ -- Spline length, inches

D -- Spline diameter, inches

B -- Tooth hardness (Brinell hardness, with units of kilograms per square millimeter)

α -- Misalignment angle, radians

F -- Mean time to failure, hours

Using the pi theorem (See Appendix A) the seven variables are put into four dimensionless groups as follows:

$$FN; \quad \frac{BD^3}{t}; \quad \frac{\ell}{D}; \alpha$$

Brinell hardness is used because it has a specific set of units. If other values, such as Rockwell C are used, they must be converted to Brinell.

To simplify the frequent use of these four parameters throughout the example to follow, two new symbols are to be used, namely θ and ϕ.

$\theta = 60FN$ and may be considered as the life of the spline in total revolutions.

$\phi = \frac{1422BD^3}{t}$ and may be considered as a "strength-to-load" ratio; with higher numbers indicating longer life for a given ℓ/D ratio and misalingment, α.

The ℓ/D ratio is large for long slender splines and small for short stubby splines. In this case the diameter was taken to be the pitch diameter.

Since F is in hours and N is in revolutions per minute, the constant 60 minutes per hour is included to make θ dimensionless. Similarly 1422 is included in ϕ to make the units of Brinell hardness campatible with the units of D and t.

The procedure now is to find the relationship that exists among the four variables from some existing theory or data. Although any one of the four variables may be used as the explicit variable, we will choose θ as the most desirable. That is,

$$\theta = f(\phi, \ell/D, \alpha). \tag{1}$$

Data

In order to find the unknown function in the above equation some spline test data was sought that included all the parameters. Some test data was available, such as from Southwest Research Institute (12),

but the values of N, ℓ, D, and t were the same for all the runs. An involute spline design procedure outlined by Raymond J. Drago (13) includes all the needed variables in chart form to establish the above functional relationship. It was decided to use this procedure since results would be more complete. A copy of this procedure from Machine Design, February 12, 1976, is included as Appendix B.

Table 1 in Appendix C was constructed by choosing several different values of L while holding all other variables constant to give different ℓ/D values. The second half of the table is constructed to give different values of ϕ while holding all other variables constant.

Table 2 in Appendix C is a more complete set of data for full depth involute splines taken from the Drago charts.

The quantities in Tables 1 and 2 such as J (geometry factor), T_F (torque factor), B_s (bearing stress, and K_L (life factor) are explained in Appendix B. Also the numerical values of α in both tables were obtained by taking the mid-range value from those given in Appendix B for light, moderate, and heavy misalignment.

Establishing the Relationship Among the Dimensionless Variables

Graphical. The curves shown in Figures 1 and 2 show how θ varies with ϕ and ℓ/D. When the two curves are compared it is noted that they are similar if not identical. This implies that the two variables, ℓ/D and ϕ, may be combined into a single dimensionless variable to reduce the number of variables to 3 if so desired. This provides an extra measure of flexibility in using the dimensionless variables. Reduction to 3 variables reduces still further the amount of test data needed to establish a mathematical life model for a new type of spline; but keeping

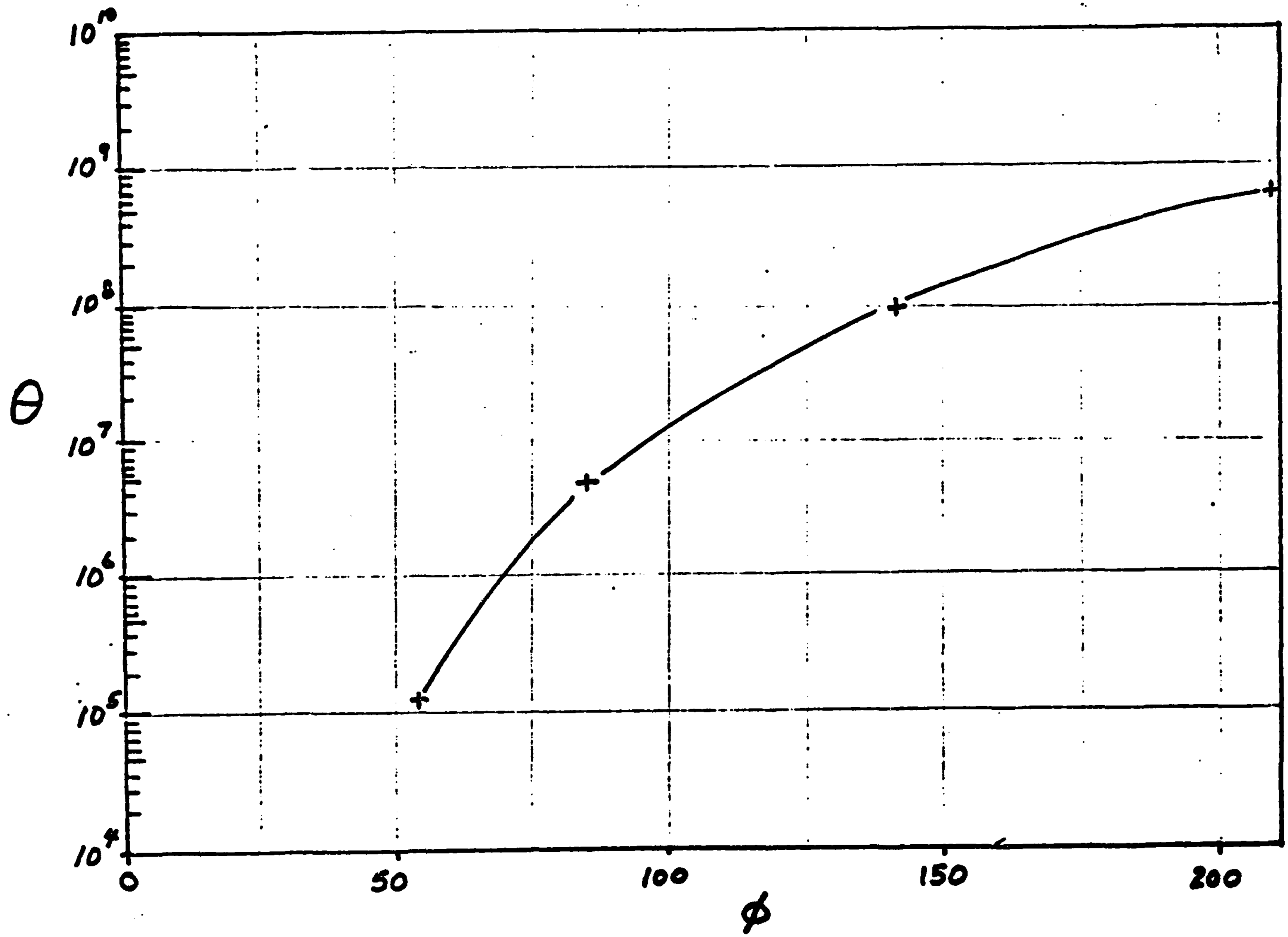

Figure 1. θ vs. ϕ.

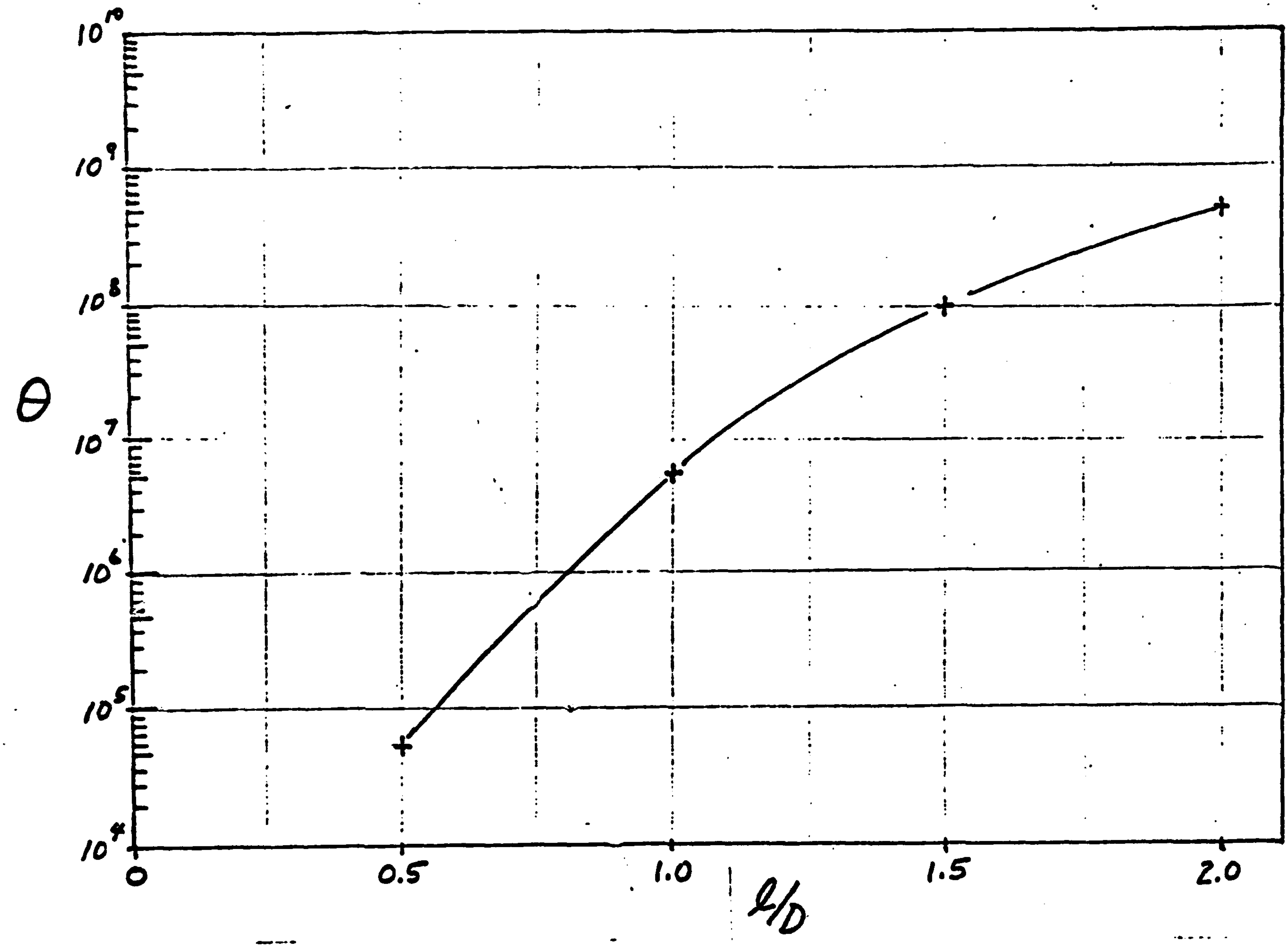

Figure 2. θ vs. ℓ/D.

ℓ/D as a separate variable gives the designer a special flexibility factor for selecting spline geometry for maximum life.

Figure 3 is a set of curves of log θ vs. log $(\frac{\phi\ell}{D})$ for different values of α. This represents the relationship among the variables in graphical form. The physical significance of $\frac{\phi\ell}{D}$ is that it represents a ratio of tooth hardness to bearing pressure for the particular spline and its application.

Regression Analysis. Regression analysis was used to determine the exact functional relationship for θ as a function of ℓ/D, ϕ, and α using the 33 "data" points previously discussed. Several linear regression models were investigated.

The first approach used a stepwise regression technique to predict θ with the variables ℓ/D, ϕ, α, their squares, and their cross products. In stepwise regression, variables are added to the model one at a time until no further improvement can be found. The variable with the highest correlation with the dependent variable is added first. Other variables are chosen on the basis of their ability to produce a significant F statistic. The F statistic is indicative of the predictive power of the model being tested. It also reflects the goodness of the fit. The higher the F value, the better the model describes the data. The term significant means that the F value being examined has a very small probability (<.05) of occurring by chance.

The stepwise regression procedure yielded a model of the form $\theta = a_0 + a_1 \alpha + a_2 \alpha^2$. For this model $R^2 = .26$. R^2 indicates the proportion of the variability in θ explained (or accounted for) by the

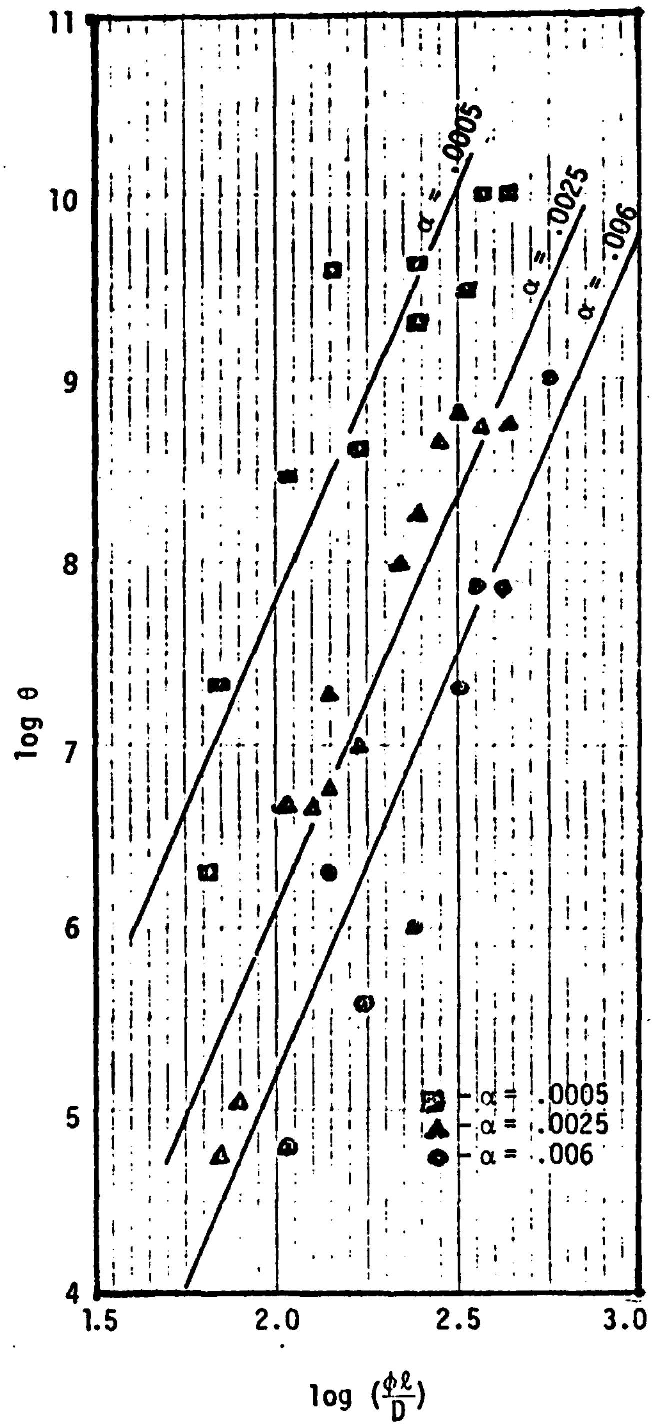

Figure 3. log 0 vs. log ($\frac{\phi\ell}{D}$).

current regression model. Since 74% of the variance in θ was unexplained by the above model, it was deemed unsatisfactory and another approach was initiated.

Figure 3, a plot of $\log \theta$ vs. $\log (\ell/D \cdot \phi)$ for each of the three values of α indicated a strong linear relationship. A regression model of the form $\log \phi = a_0 + a_1 \log (\ell/D \cdot \phi)$ offered support for this notion as $R^2 = .81$ for the case $\alpha = .0005$, $R^2 = .82$ for $\alpha = .0060$ and $R^2 = .93$ for $\alpha = .0025$.

A single model including the contribution of α was then investigated. This model was

$$\log \theta = -9.15 + 4.56 \log (\ell/D \cdot \phi) - 2.36 \log \alpha.$$

The resulting R^2 value was .89. Several other models were investigated, but each of the other models had an R^2 less than 89%. The above model was chosen as the best predictor of $\log \theta$ since it explains 89% of the variance in $\log \theta$.

Taking the antilog of both sides of this model, we have

$$\theta = 7.08 \times 10^{-10} (\ell/D \cdot \phi)^{4.56} \alpha^{-2.36}. \qquad (2)$$

This equation was used to generate 33 predicted values of θ and comparisons were made with the actual data points. Figure 4 shows a comparison of the values of θ predicted by equation (2) vs. the actual data point values of θ for the 33 data points. Figure 4 along with the value of R^2 = 89% demonstrates that MTTF for this spline case can be predicted by developing a mathematical model using dimensionless groupings.

It should be pointed out that some of the scatter in the "data" is due to the fact that each value of θ is a result of reading values from 4 charts in sequence so that there can be considerable accumulation of

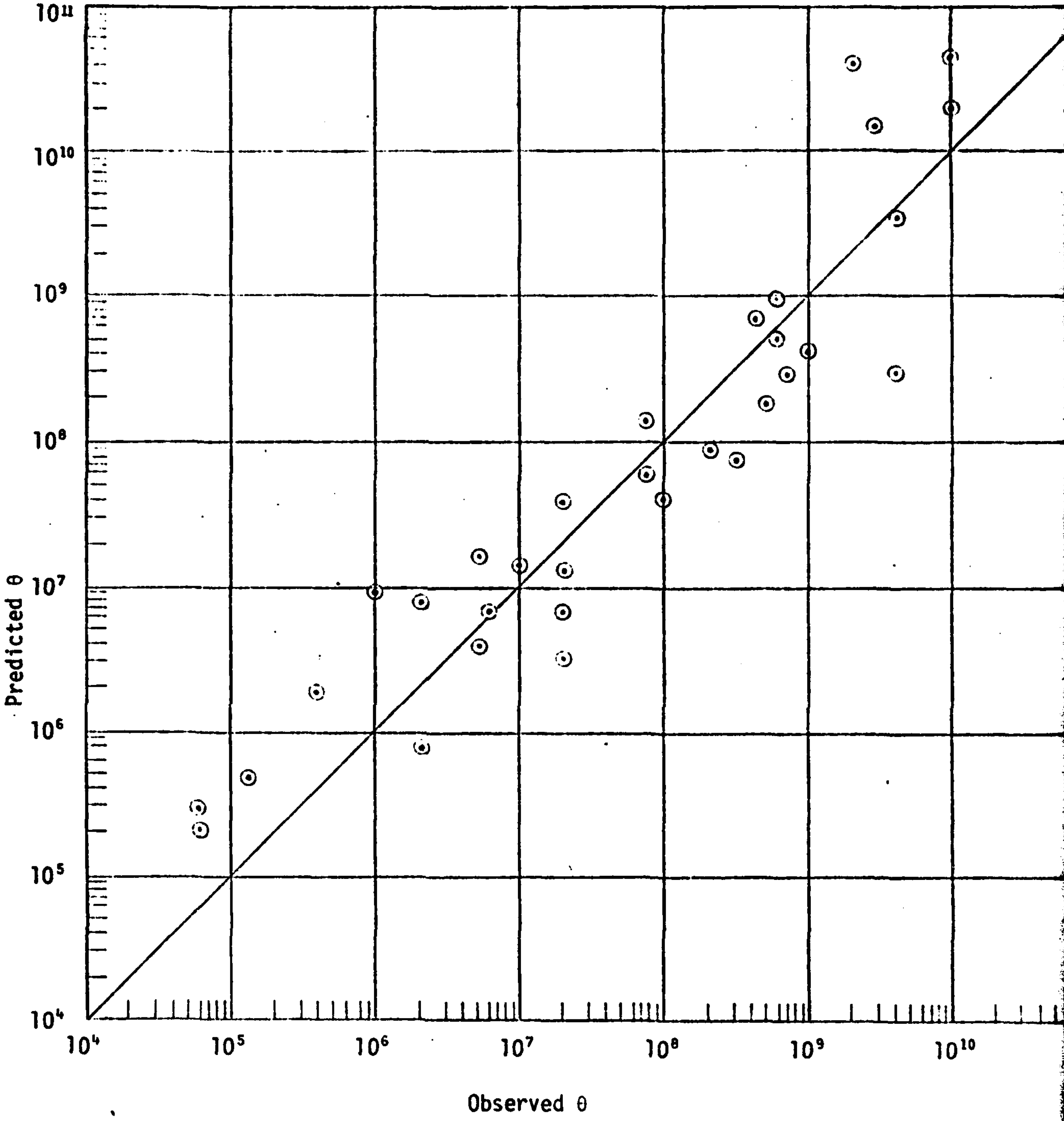

Figure 4. Comparison of θ predicted by equation (2) to actual data point values.

error due to reading of the charts. This scatter is not considered detrimental to the present study since any actual experimental data would exhibit scatter due to experimental error, variation in material properties, etc. Another point of note is the fact that the Drago method represents a very elementary mathematical model and predicted values may differ considerably from actual experimental values; therefore, the mathematical model obtained using dimensional analysis and regression analysis will predict MTTF with an accuracy that is no better than the accuracy of the data on which it is based, (i.e., the Drago method).

CONCLUSIONS

The preceding example demonstrates that dimensional analysis is an important tool for use in the prediction of mechanical reliability and should be implemented in all future test programs. Through the use of dimensionless groupings of variables, a predicting equation for mean time to failure as a function of 7 independent variables was fitted to 33 "data" points. The primary advantages of the use of dimensional analysis are:

1. The number of variables in the problem is reduced, thus reducing data requirements and simplifying the application of regression analysis.

2. Dimensional analysis can be very useful in planning experimental work to get maximum information from a minimum amount of raw data since only the magnitude of each dimensionless group needs to have a range of values and every individual variable does not need to be changed. Also, the most important or most influential variables may be isolated easier when the variables are in dimensionless groups. By performing a dimensional analysis before an experiment is planned, the testing can be done more economically. Since one dimensionless group can be varied by varying only one of the variables in the group, the experimenter can choose to control or vary the dimensionless group by varying or controlling the variables that are most easily varied or controlled.

3. A mathematical model written in the form of dimensionless groups of variables for a specific design or type of item can be more readily adapted to a new design of the item. For example, it is likely that a correlation for involute splines could be revised for circular splines in an easier manner if the correlation is in the form of dimensionless groups of variables.

In addition, the use of dimensional analysis allows prediction of reliability for certain operating conditions for which data is unavailable. For example, suppose in the process of designing a spline that is to operate at a speed of 4000 rpm, we find that the data is available only for a speed of 2000 rpm. If a predicting equation in dimensionless form is fitted to the 2000 rpm data, then, in the absence of 4000 rpm data, the predicting equation can be used to predict reliability of splines at 4000 rpm.

RECOMMENDATIONS FOR FUTURE WORK

1. There are strong indications that dimensional analysis could be used in developing improved methods of accelerated testing. Further work should be done to determine the feasibility of using dimensionless variables to relate the results of accelerated testing to actual use conditions.

2. The applicability of dimensional analysis to the prediction of the reliability of variety of mechanical components should be investigated. Such an investigation would include further work on the prediction of reliability for splines as well as devices such as gears, pumps, etc. It is expected that such an investigation would provide input for planning and analysis of current and future mechanical reliability test programs.

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APPENDICES

APPENDIX A

Determination of Dimensionless Variables for Spline Example

We wish to form a complete set of dimensionless variables for the case of full-depth involute splines as discussed in the section Forming the Dimensionless Groups. The method outlined in Reference (7) will be used. The pertinent variables along with their dimensions are:

Variable	Symbol	Dimension
Mean Time to Failure	F	$[T]$
Speed	N	$[T^{-1}]$
Torque	t	$[ML^2T^{-2}]$
Length	ℓ	$[L]$
Diameter	D	$[L]$
Hardness	B	$[ML^{-1}T^{-2}]$
Angular Misalignment	α	Dimensionless

Since the misalignment is already dimensionless, we will deal with the first 6 variables only. Any dimensionless grouping of these 6 variables will be of the form:

$$\Pi = F^a N^b t^c \ell^d D^e B^f$$

where Π represents a dimensionless group (sometimes called a "pi-term"). The values of the exponents a, b, c, etc., will be determined such that we have a complete set of dimensionless variables formed from the 6 variables. The dimension equation for Π is

$$[M]^0 [L]^0 [T]^0 = [T]^a [T^{-1}]^b [ML^1 T^{-2}]^c [L]^d [L]^e [ML^{-1} T^{-2}]^f.$$

The above equation may also be written as

$$[M]^0 [L]^0 [T]^0 = [M]^{c+f} [L]^{2c+d+e-f} [T]^{a-b-2c-2f};$$

therefore, we can write the following 3 equations:

$$c+f = 0$$

$$2c+d+e-f = 0$$

$$a-b-2c-2f = 0$$

The matrix of the coefficients of the 6 unknown in the above equation is:

$$\begin{matrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \\ 1 & -1 & -2 & 0 & 0 & -2 \end{matrix}$$

The rank of the above matrix is 3; therefore, the number of pi-terms to be formed from the 6 variables is 6 - 3 = 3.

Noting that we have 3 equations and 6 unknowns, we must choose values for 3 of the unknowns and solve for the remaining 3 in determining each pi-term. For the first pi-term, we choose $a = 1$ (This insures that F will appear in Π_1), $c = d = 0$. The 3 simultaneous equations now become:

$$f = 0$$

$$e-f = 0$$

$$1-b-2f = 0$$

The solution of this set of equations yields $f = 0$, $e = 0$, $b = 1$ so that

$$\Pi_1 = F^1\ N^1\ t^0\ \ell^0\ D^0\ B^0$$

$$\Pi_1 = FN$$

We want the dependent variable, F, to appear in one pi-term only; therefore, in determining Π_2, we let a = 0. We also let c = 0, d = 1. The set of equations becomes:

$$f = 0$$

$$1 + e - f = 0$$

$$b - 2f = 0,$$

the solution of which is f = 0, b = 0, e = -1. Therefore,

$$\Pi_2 = F^0\ N^0\ t^0\ \ell^1\ D^{-1}\ B^0$$

$$\Pi_2 = \ell/D$$

Finally, let a = d = 0, f = 1 so that the set of equations becomes

$$c + 1 = 0$$

$$2c + e - 1 = 0$$

$$-b - 2c - 2 = 0$$

The above set of equations has the solution c = -1, e = 3, b = 0, yielding

$$\Pi_3 = F^0\ N^0\ t^{-1}\ \ell^0\ D^3\ \ B^1$$

$$\Pi_3 = \frac{BD^3}{t}$$

These 3 pi-terms along with α which is already dimensionless are a complete set of 4 dimensionless variables formed from the variables pertinent to the spline case:

$$\Pi_1 = FN$$

$$\Pi_2 = \ell/D$$

$$\Pi_3 = \frac{BD^3}{t}$$

$$\Pi_4 = \alpha$$

The above dimensionless variables form a complete set because each dimensionless variable in the set is independent of the others in the sense that no dimensionless variable in the set is a product of powers of the other dimensionless variable. Although innumerable other dimensionless variables can be formed, any other dimensionless variable is a product of powers of the dimensionless variables in the set.

Obviously, there is an infinite number of complete sets of dimensionless variables other than the set developed above. Selection of the best complete set depends on the way in which the model is to be used and the experience of the engineer. Any knowledge available concerning the phenomenon enhances the ability of the engineer to select the best complete set.

APPENDIX B

Rating the Load Capacity of

Much work has been done to standardize spline geometries, but little has been done to standardize load-capacity ratings. Thus, spline performance is difficult to predict with conventional design procedures. Now, with a new set of charts, service life can be determined quickly.

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INVOLUTE SPLINES connect rotating parts in applications as diverse as helicopter rotors and machine-tool gearboxes. But too often, each application uses its own special design. There is very little literature on how to determine the interchangeability of a spline among different operating conditions.

As a result, a new spline application must be analyzed in depth to determine how long a particular design will last or, conversely, which design best fits the functional requirements. Now, with a newly developed set of design charts, a spline can be checked quickly to determine its life under specific operating conditions. Or, the performance requirements can be used to find a spline design that will last for the desired life.

The charts are based on equations common to the industries that use involute splines. They represent "average" operating conditions, and they apply to almost any spline application.

The prime factors to consider in spline design are the compressive (surface bearing) stress S_c and the shear (tooth) stress S_s. The equations for these stresses are

$$S_c = (2T/D)(1/NLH)(K_m) \le S_{ca}K_L/K_aK_T$$

$$S_s = (2T/D)(1/DNLt)(K_m) \le S_{sa}K_l/K_a$$

These equations (with N, H, and t converted to functions of diametral pitch) are the basis for the design charts presented here. The only approximation involved in the charts is in the use of "nominal" tooth thickness in the shear stress equation. For most designs, this approximation introduces a maximum error of less than 5%.

Before getting into a discussion of how to use the charts, it is important to understand how splines work, how they fail, and how various processes help prolong life. This understanding makes the load factors (K_m and K_a) easier to estimate.

Types of Spline Joints

Generally, spline joints fall into two categories: fixed and working. In a working joint, the mating members are free to move relative to each other. But in a fixed joint, both ends of the spline are clamped and piloted to prevent relative motion. In addition, many intermediate (semiworking or semi-fixed) spline joints are possible. For example, a spline with loose pilots at each end or with O-rings to retain a grease pack, while not fully fixed, cannot be considered fully working either because relative motion is restricted to some degree.

A working spline generally joins two independently supported shafts that may also be slightly offset axially (from tolerance buildups). This type of spline accommodates the offset easily without excessive loss in basic load capacity. Fixed splines may also join independently supported shafts, but the shafts must be aligned more accurately. Most frequently, however, fixed splines mount precision gears to shafts, particularly in high-load, high-speed, lightweight applications.

Usually, the surface load capacity of a fixed spline is considerably higher than that of a working spline. In a working spline, relative motion between mating surfaces induces a high wear rate, even at relatively low bearing stresses. But in a fixed joint, restricted motion results in a low wear rate, making surface fatigue the primary failure mode.

Spline Failure Modes

Spline joints "fail" in any number of different ways, but many failure modes (such as fretting corrosion) cannot be analyzed directly. Fortunately, physical failures can be analyzed, and they are generally divided into four classes. Two classes, shaft shear (torsional shear failure of the shaft section under the spline teeth) and bursting of an internal spline (from radial and centrifugal loads),

Involute Splines

are primarily shaft related and are not treated here because they do not occur often.

The other two tooth-related failure modes are much more common. These modes are tooth shear (actual shearing off of a tooth projection from the shaft) and surface distress of the teeth (from wear or surface fatigue). In the shear mode, a spline immediately ceases to transmit power, while in the wear mode it degrades gradually before complete failure. Wear failure can only be detected during periodic inspection.

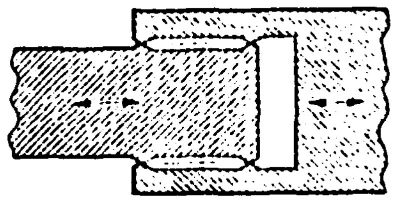
Surface distress is by far the most common failure mode encountered in spline joints. This failure may take the form of wear (material removed from mating surfaces) or spalling of mating surfaces from fatigue. Frequently, over a long period of time, these two types of failure combine, with fatigue resulting from a worn surface. On the other hand, depending on spline configuration, the joint may "wear-in" to remove a potential failure zone. Here, highly loaded local areas caused by high spots, slight misalignment, or tooth index errors may wear to relieve the load and distribute it over a larger area, thus prolonging life.

One of the most significant factors causing wear in a spline joint is misalignment, which causes high bearing stresses at the contact points. Normally, with a well aligned shaft, wear can be prevented by lengthening a spline to reduce bearing stress, thus lowering wear rate. However, if wear is caused by excessive misalignment, lengthening the spline is not likely to reduce wear rate. Better solutions are to reduce length and increase diameter, or perhaps crown the teeth of the external member. Of course, the best solution is to eliminate misalignment altogether; however, this fix may not be practical from a design standpoint (the misalignment may be the very reason why the spline is used). Generally, though, if substantial misalignment is inherent in the application, a crowned spline, a large-diameter short-length spline, or a more complicated coupling should be custom designed for the application.

One way to overcome slight misalignment (particularly in high speed, high-load, precision gearboxes) uses a "quill" shaft between the driving and driven shafts. Basically, a quill shaft is an extra shaft, splined at both ends, that floats between the other two shafts. This free-floating capability allows the quill shaft to self-align to some degree to accommodate the mismatch between the driving and driven members. But quill shafts are difficult to lubricate and position. And, if misalignment is large, a significant bending moment may be applied to the quill shaft, so it behaves somewhat like a rotating fatigue-test specimen. Crowned teeth add to the misalignment capability of a quill shaft, especially in high-speed, high-load applications in aircraft.

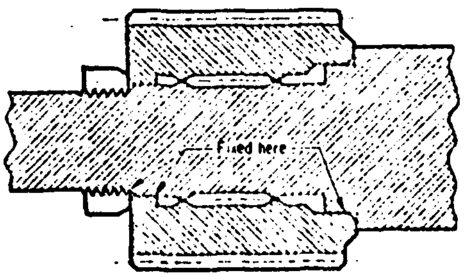
Shear failure, as a primary mode, is relatively

Working Spline



In a working spline joint, the mating parts are *free to move relative to each other.* This type of spline usually connects two independently supported shafts that also may be slightly offset *axially.* It accommodates the offset with little loss in load bearing capacity. Wear is the predominant failure mode.

Fixed Spline



A fixed spline joint restricts relative motion between mating parts. This type of spline usually mounts precision gears to shafts in high-load, high-speed, *lightweight applications.* Wear combined with surface fatigue is the primary failure mode.

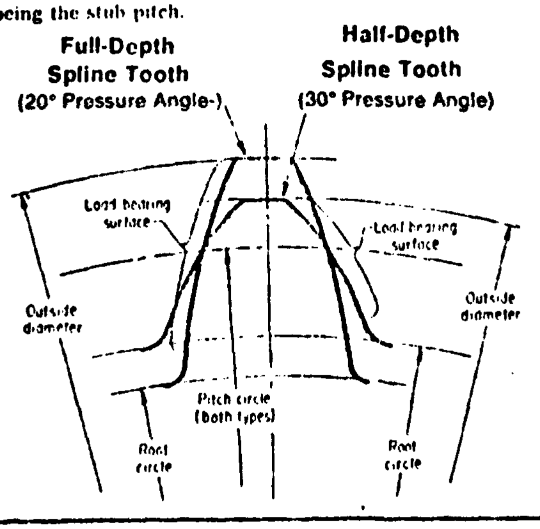
rare in most spline applications. This failure usually occurs as a secondary mode after the teeth have worn excessively thin or as a result of severe overload or impact loading. However, shear can be a primary mode in a spline joint subjected to severe impact loading throughout its life. In this case, shear analysis should be emphasized because the spline may fail in shear before it has a chance to wear in. Therefore, even though shear may not be a determining factor in the design of a spline joint, shear stresses should always be checked as a matter of course.

Full-Depth vs. Half-Depth Teeth

Both full and half-depth splines can be analyzed with these charts. Most splines in use are half-depth because they have a number of advantages over full-depth splines. First, because tooth height generally determines the amount of backup material required behind the teeth, the half-depth spline provides a more compact, and often lighter, overall design. Second, the relatively high teeth on a full-depth spline are somewhat more sensitive to misalignment. In addition to these operational considerations, several manufacturing methods, such as broaching or roll forming, are easier to perform on the smaller half-depth spline teeth.

In spite of these considerations, full-depth splines have about double the surface (bearing) load capacity of equivalent half-depth splines. In applications where this increased capacity is needed, it is worth the effort to overcome the inherent disadvantages of the full-depth design. A typical application for full-depth splines is on the rotor shafts of some large helicopters; joints that are generally fixed so that other variables can be controlled. Another application is in high-speed, "gear" couplings that use full-depth (often crowned) teeth to connect rotating shafts.

The standard pressure angle for half-depth splines is 30°. The pressure angle of full-depth splines commonly is 20°, however, it sometimes must be varied according to the particular design to avoid pointed teeth. The drawing shows the differences in tooth size and bearing area between full and half-depth splines. The addendum of a full-depth spline equals $1/Pd$, while that of a half-depth spline equals $1/Ps$, where Pd = diametral pitch (ratio of number of teeth to pitch diameter) and Ps = "stub" pitch ($2Pd$). The pitch of a spline commonly is given as a ratio (16/32, 20/40, and so on) with the numerator being the diametral pitch and the denominator being the stub pitch.



Coatings and Surface Treatment

Wear is often caused by poor lubrication. Som times, despite the most elaborate lubrication sy tem, the only way to prevent wear is to apply coating or surface treatment.

The most common coatings are soft ones li black oxide, phosphate, nylon, silver, and copp and various hard coatings such as flame or plasm sprayed materials. The soft coatings aid initi wear-in and also lubricate the surface with a plia film during operation. The hard coatings approa the wear problem from the opposite end of t spectrum; that is, they produce a surface that sists wear by virtue of extreme hardness.

If the surface bearing stresses are low (less th 3,000 psi), a nylon coating is often the best w to prevent wear and damp the spline joint. The co ing, a few thousandths of an inch thick, is appli to the entire spline surface (generally on the e ternal member). Usually, to allow room for t thickness of the nylon and still provide proper f the cut size of the spline must be reduced. T other coatings are thin enough (on the order a few thousandths of an inch) so that no modific tion is required.

Other alternatives to improve spline load capacit include case hardening processes such as carb rizing and nitriding. These processes generally pr duce a very hard (R_c 58 to 64) wear resista surface. Unfortunately, they also increase the co of the parts because special operations are neede to control or correct heat treatment distortion

EXAMPLES

Problem 1: Consider a fixed, half-depth spline tr mitting torque T = 1,000 lb-in. and operating under l shock and misalignment (less than 0.001 in./in.). spline is made of AISI 9310 case carburized steel w a surface hardness of R_c 60 to 64. Dimensions are 0.30 in. and D = 0.75 in. Find the bearing and s stresses and the projected life.

1. On Chart 1, at L = 0.30 in. and D = 0.75 in., J = 9.5.
2. On Chart 2 (half-depth bearing stress plots, li misalignment), at J = 9.5, find K = 9.5 for bea stress.
3. On Chart 2 (shear stress plots, light misalignme at J = 9.5, find K = 6.0 for shear stress.
4. Calculate S_{c4} = (1000) (9.5) = 9,500 psi.
5. Calculate S_{s4} = (1000) (6.0) = 6,000 psi.
6. On Chart 3A, for S_{c4} = 9,500 psi, K_e = 1.0 (li shock), and R_c 60 carburized steel, find bea stress life factor K_L = 1.30.
7. On Chart 3B, for S_{s4} = 6,000 psi, K_e = 1.0, R_c 60 carburized steel, find shear stress life fa K_L = 0.125.
8. On Chart 4, at K_L = 1.30 for a fixed spline, bearing stress life = 7.4 × 10^9 revolutions.
9. On Chart 4 again, at K_L = 0.125, find shear st life = ∞.
10. Probable life is 7.4 × 10^9 revolutions.

Although surface-hardened parts have a high wear life, they may perform poorly if not made accurately. These parts do not "wear-in" very well and may destroy themselves if each tooth does not carry its share of the load. On smaller splines, lapping is an inexpensive and often effective way to finish hardened surfaces. This process improves the finish and corrects minor errors at considerably lower cost than grinding.

Lubrication

Because splines are often outside an enclosure, a pressurized oil system may not be available to lubricate the joint. Therefore, grease is a common spline lubricant. Generally, the spline joint is packed full of grease and a retention method devised so that the grease does not centrifuge out of the joint.

One drawback to grease lubrication is that wear particles are trapped in the grease. After a time, the mixture becomes abrasive and may accelerate wear. For this reason, grease lubricated working splines should be regreased periodically, and the old grease (and wear particles) purged rather than merely replenished. One exception to this rule is the case of a fully locked or fixed spline. If relative motion is severely restricted, the initial grease pack may last for hundreds or even thousands of hours.

If the spline is internal and the gearbox is oil lubricated, the joint may be lubricated by oil splash or direct oil jet. In either case, the oil flow both lubricates and removes wear particles, thus reducing wear.

Another method of lubrication that has been used with some success directs oil down the center of a hollow shaft to radial holes in the spline. The oil flows by centrifugal force through the spline and out through external metering holes. The holes should be large enough to prevent clogging yet small enough to "flood" the tooth area. This method is an effective means of cooling and lubricating the joint and flushing out sludge.

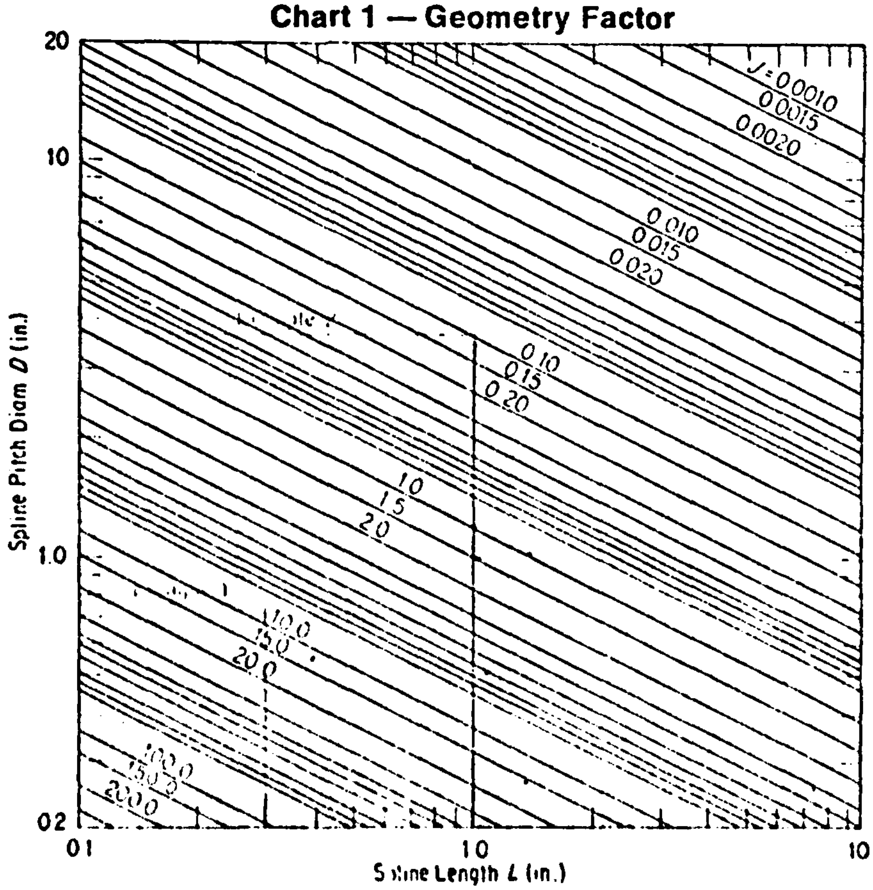
Coatings and lubricants generally work in concert. For example, black oxide enhances the oil retention or oiliness of spline teeth, and silver plate acts like an extreme pressure film during break-in and operation. In some cases, however, lubrication is superfluous, as with some forms of nylon coatings (these coatings operate best when unlubricated).

Rating Spline Load and Life

With this basic understanding of the factors that affect spline life, the use of the design charts for rating load and life will be more clear. The charts can be used in either of two cases. First, if spline material, geometry, and operating conditions are known, the load capacity and potential life of the joint can be determined. Second, if the desired life, loading conditions, and material are known, the geometry can be found.

Two factors must be determined before the

Chart 1 — Geometry Factor



analysis can begin: the load distribution factor K_m and the overload factor K_o. K_m is a function of the misalignment present in the joint. Since misalignment can have a significant effect on the load capacity, care should be taken to choose as realistic a value as possible for K_m. Typically, for light misalignment (0.000 to 0.001 in./in.), $K_m = 1.0$; for moderate misalignment (0.001 to 0.004 in./in.), $K_m = 2.0$; and for heavy misalignment (0.004 to 0.008 in./in.), $K_m = 3.0$.

The actual peak load applied to a spline may be considerably higher than the steady-state design load. Factor K_o adjusts the load rating to account for any shock loads on the driven and driving members. Typical values for K_o are listed in the table near Chart 3.

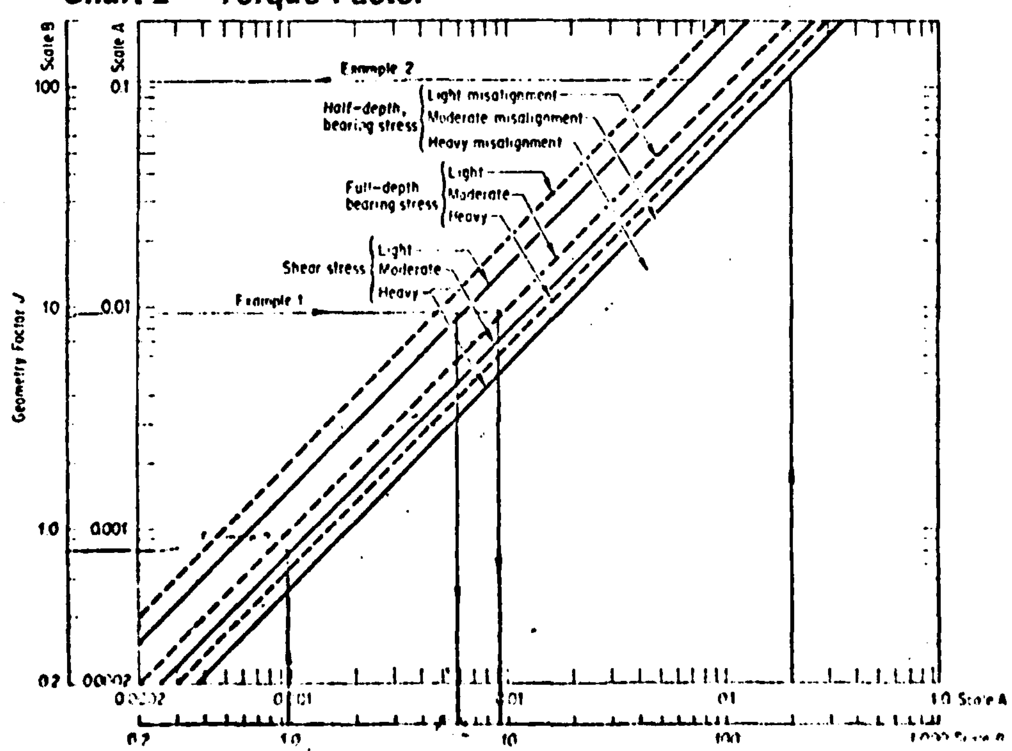
The procedure when load and life must be determined from material and geometry is:

1. Chart 1: Find geometry factor J at known pitch diameter D and length L.
2. Chart 2: On the proper curve (relating misalignment, stress, and spline design), find torque factor K at known J and known (or estimated) K_m.
3. Calculate allowable bearing stress S_{cA} and allowable shear stress S_{sA} from $S = TK$.
4. Charts 3A and 3B: Find two life factors K_L at known values of S_{cA} and S_{sA} for the given material.
5. Chart 4: At known values of K_L, find the projected spline life. Use lower life as projection.

Problem 2: Consider a spline made of 300 Bhn steel that must transmit a torque $T = 8{,}000$ lb-in. under conditions of moderate shock and misalignment. Required life is 10^7 revolutions. Determine the spline dimensions for this application.

1. On Chart 4, at life = 10^7 revolutions, find $K_L = 0.3$ for shear stress and $K_L = 0.9$ for bearing stress.
2. On chart 3A, at $K_L = 0.9$, 300 Bhn steel, and $K_o = 1.5$ (moderate shock), find $S_{cA} = 1{,}700$ psi.
3. On Chart 3B, at $K_L = 0.3$, 300 Bhn steel, and $K_o = 1.5$, find $S_{sA} = 8{,}000$ psi.
4. Calculate bearing stress torque factor $K = 1700/8000 = 0.21$.
5. Calculate shear stress torque factor $K = 8000/8000 = 1.00$.
6. On chart 2 (half-depth bearing stress plots, medium misalignment), at $K = 0.21$, find $J = 0.108$ for bearing stress.
7. On Chart 2 (shear stress plots, medium misalignment, at $K = 1.00$, find $J = 0.80$ for shear stress.
8. Since J for bearing stress is lower than J for shear stress, bearing stress is the more important factor. Therefore, any combination of L and D that meets along the $J = 0.108$ line on Chart 1 will satisfy the stress and life requirements. The choice of L and D should be based on the available space, keeping Ratio L/D as small as practicable. For example, a 3.6 in. diameter spline, about 1.0 in. long will do the job here.

Chart 2 — Torque Factor



For determining geometry from desired life and material, the procedure is:

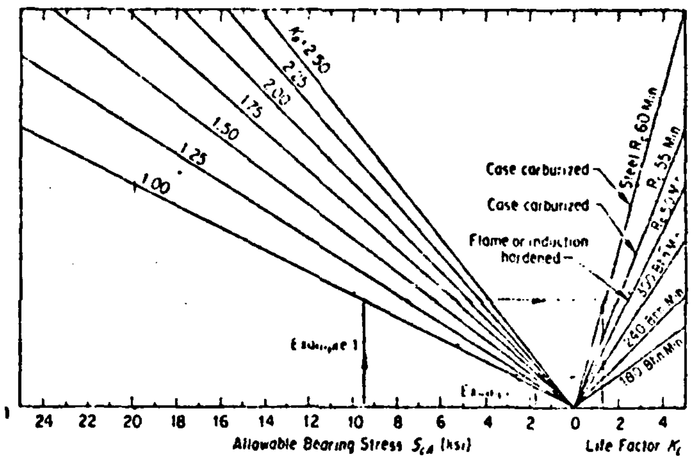
1. Chart 4: At desired life, find life factor K_L.
2. Charts 3A and 3B: At known K_L, find allowable bearing and shear stresses S_{cA} and S_{sA}.
3. Calculate a torque factor K for both bearing and shear stress from $K = S/T$.
4. Chart 2: On the relevant curves, at known K and known (or estimated) K_m, find a geometry factor J for both bearing and shear stress.
5. Chart 1: Use the smaller J from Step 4 to find length L and pitch diameter D.

Both procedures determine the bearing stress *and* the shear stress. Generally, bearing stress limits long-term life while shear stress limits the impact or shock load capacity. □

Nomenclature

- D = Spline pitch diam, in.
- H = Tooth height, in.
- K_L = Life factor
- K_m = Load distribution factor
- K_o = Overload factor
- K_T = Spline type factor
- L = Spline length, in.
- N = Number of teeth
- S_c = Compressive (surface bearing) stress, psi
- S_{cA} = Allowable bearing stress, psi
- S_s = Shear (tooth) stress, psi
- S_{sA} = Allowable shear stress, psi
- T = Torque, lb-in.
- t = Tooth thickness, in.

Chart 3A — Bearing Stress Life Factor



How Overload Factor Reflects Shock Loads

Overload Factor K_o		—Shock on Driven Member— Light	Moderate	Heavy
Shock on	Light	1.00	1.25	1.75
Driving	Moderate	1.25	1.50	2.25
Member	Heavy	1.50	1.75	2.50

Chart 3B — Shear Stress Life Factor

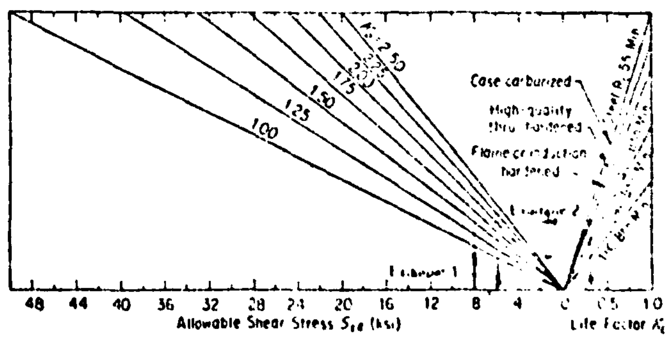
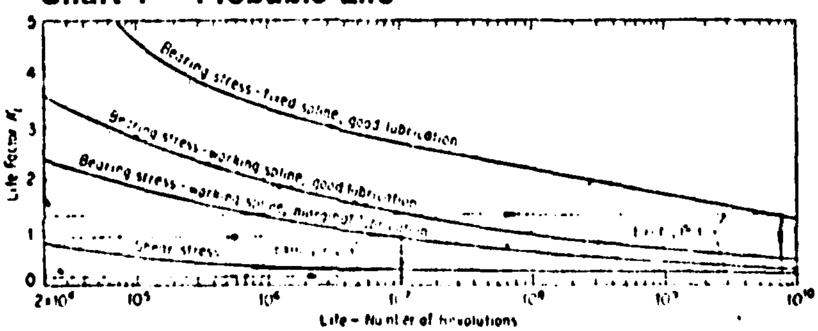


Chart 4 — Probable Life



APPENDIX C. TABLE 1

L	D	B	t	α	J	T_F	B_s	K_L	θ	L/D	ϕ	ϕL/D
0.5	1.0	300	3000	.0025	3.0	3.0	9000	3.0	5.6×10^4	0.5	142	71
1.0	1.0	300	3000	.0025	1.5	1.5	4500	1.5	6.0×10^6	1.0	142	142
1.5	1.0	300	3000	.0025	1.0	1.0	3000	1.0	1.0×10^8	1.5	142	213
2.0	1.0	300	3000	.0025	0.75	0.75	2250	0.75	5.0×10^8	2.0	142	284
1.5	1.0	300	2000	.0025	1.0	1.0	2000	0.67	7.0×10^8	1.5	213	320
1.5	1.0	300	3000	.0025	1.0	1.0	3000	1.0	1.0×10^8	1.5	142	213
1.5	1.0	300	5000	.0025	1.0	1.0	5000	1.67	5.0×10^6	1.5	85	128
1.5	1.0	300	8000	.0025	1.0	1.0	8000	2.67	1.2×10^5	1.5	53	80

APPENDIX C. TABLE 2

L	D	B	t	α	J	T_F	B_s	K_L	θ	L/D	φ	φL/D
1.0	0.5	300	1,000	.0025	6.00	6.20	6,200	1.50	5.0×10^6	2.0	53	106
1.0	2.0	180	6,000	.0025	0.38	0.39	2,340	1.40	1.0×10^7	0.5	341	171
2.0	0.5	600	3,000	.0025	3.00	3.0	9,000	1.20	2.0×10^7	4.0	35.5	142
2.0	3.0	500	30,000	.0025	.0833	.0833	2,500	0.70	6.0×10^8	0.667	640	427
1.0	4.0	240	15,000	.0025	.094	.094	1,410	0.70	6.0×10^8	0.25	1,456	364
1.0	3.0	300	15,000	.0025	.167	.167	2,500	0.85	2.0×10^8	0.33	768	256
1.0	0.5	180	1,000	.0005	6.00	2.8	2,800	1.8	2.0×10^6	2.00	32	64
0.5	1.0	300	3,000	.0005	3.0	1.4	4,200	1.2	2.0×10^7	0.5	142	71
1.0	0.5	300	1,000	.0005	6.0	3.0	3,000	0.8	3.0×10^8	2.0	53	106
1.0	2.0	180	6,000	.0005	0.38	0.2	1,200	0.75	4.0×10^8	0.5	341	171
2.0	0.5	600	3,000	.0005	3.0	1.4	4,200	0.5	4.0×10^9	4.0	35.5	142
2.0	3.0	500	30,000	.0005	.0833	.0417	1,250	0.35	1.0×10^{10}	0.67	640	427
1.0	4.0	240	15,000	.0005	.094	.047	705	0.35	1.0×10^{10}	0.25	1,456	364
1.0	3.0	300	15,000	.0005	.167	.083	1,250	0.43	2.0×10^9	0.33	768	256
1.5	2.5	500	20,000	.0005	.160	.080	1,600	0.40	3.0×10^9	0.60	555	333
2.5	1.5	300	10,000	.0005	.270	.135	1,350	0.50	4.0×10^9	1.67	144	240
1.0	0.5	300	1,000	.006	6.0	9.0	9,000	3.0	6.0×10^4	2.0	53	106
1.0	2.0	180	6,000	.006	0.38	0.58	3,480	2.2	3.6×10^5	0.5	341	171
2.0	0.5	600	3,000	.006	3.0	4.4	13,200	1.8	2.0×10^6	4.0	35.5	142
2.0	2.0	500	10,000	.006	0.19	0.3	3,000	0.6	1.0×10^9	1.0	569	569
2.0	3.0	500	30,000	.006	.0833	.125	3,750	1.05	7.0×10^7	0.67	640	427
1.0	4.0	240	15,000	.006	.094	.141	2,115	1.05	7.0×10^7	0.25	1,456	364
1.0	3.0	300	15,000	.006	.167	.250	3,750	1.28	2.0×10^7	0.25	768	256
1.5	2.5	500	20,000	.006	.160	.240	4,800	1.20	2.0×10^7	0.60	555	333
2.5	1.5	300	10,000	.006	.270	.405	4,050	2.0	1.0×10^6	1.67	144	240